JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 25, 710–716 (1969)

Correction and Addendum to "Summing Certain Number Theoretic Series Arising in the Sieve"

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In [1] the error terms in the two theorems must be modified. Fortunately the modification is of no consequence in the use of the theorems. The change is needed because

$$\sum_{k \leqslant z} \frac{1}{k} = \log z + 0(1)$$

is not uniformly valid for $0 < z < 1$ (when the sum becomes null) and must be replaced by

$$\sum_{k \leqslant z} \frac{1}{k} = \log z + \gamma + 0\left(\frac{1}{z}\right), \tag{1}$$

which is valid uniformly for $z > 0$.

The correct conclusion for Theorem 1 becomes

$$J_1(x) = \prod_{p|N} \left(1 - \frac{1}{p}\right) \prod{}' \left(1 - \frac{1}{p^2}\right) \left(1 + \frac{g(p)}{p+1}\right) \log x$$
$$+ 0\left(\log \log 3N + \frac{\tau(N)}{x}\right), \tag{2}$$

where $\tau(N)$ is the number of divisors of N. The term $\tau(N)/x$ did not appear before. For Theorem 2 the error now is

$$0\left\{\left(\log \log 3NX + \frac{\tau(N) \log NX}{x^{1/2}}\right) \log x + (\log \log 3N)^2\right\}, \tag{3}$$

where the new term is that involving $\tau(N)$.

The formula numbers containing a period such as (3.4) will all refer to [1].

CORRECTION OF THEOREM 1. The second formula below (3.4), with the use of (1) above, becomes

$$\sum_{j \leqslant x}{}' \frac{1}{j} = \sum_{i|N} \frac{\mu(i)}{i} \left(\log \frac{x}{i} + \gamma + 0\left(\frac{i}{x}\right)\right).$$

This introduces a term

$$\gamma \prod_{p|N} \left(1 - \frac{1}{p}\right) = 0(1)$$

and instead of H_3 a term $0(\tau(N)/x)$ which finally yields (2).

CORRECTION OF THEOREM 2. Here delete that part of [1] beginning below (3.8) and terminating with the bottom of p. 641. With $J_{21}(x)$ as in (3.8)

$$J_{21}(x) = \sum_{jk \leqslant x}{}' \frac{\mu^2(jk)}{jk} = \sum_{\substack{jk \leqslant x \\ (j,k)=1}}{}' \frac{\mu^2(j)\,\mu^2(k)}{jk}$$
$$= 2J_{31}(x) - J_{32}(x), \tag{4}$$

where if $y = x^{1/2}$

$$J_{31}(x) = \sum_{j \leqslant y}{}' \frac{\mu^2(j)}{j} \sum_{\substack{k \leqslant x/j \\ (k,jN)=1}} \frac{\mu^2(k)}{k}$$
$$J_{32}(x) = \sum_{j \leqslant y}{}' \frac{\mu^2(j)}{j} \sum_{\substack{k \leqslant y \\ (k,j)=1}}{}' \frac{\mu^2(k)}{k}.$$

By using (2) above with x replaced by x/j and N by Nj and $g(p) = 0$,

$$\sum_{\substack{k \leqslant x/j \\ (k,jN)=1}} \frac{\mu^2(k)}{k} = \prod \left(1 - \frac{1}{p^2}\right) \prod_{p|Nj} \left(\frac{p}{1+p}\right) \log \frac{x}{j}$$
$$+ 0\left(\log\log 3Nj + \frac{\tau(N)\,\tau(j)\,j}{x}\right)$$

since $\tau(Nj) = \tau(N)\,\tau(j)$ because $(N, j) = 1$. Hence

$$J_{31}(x) = \prod_{p|N} \left(1 - \frac{1}{p}\right) \prod{}' \left(1 - \frac{1}{p^2}\right) \sum_{j \leqslant y}{}' \frac{\mu^2(j)}{j} \prod_{p|j} \left(\frac{p}{p+1}\right) \log \frac{x}{j}$$
$$+ 0\left\{(\log\log 3Nx) \log x + \frac{\tau(N) \log x}{x^{1/2}}\right\},$$

where use is made of (3.6). Using $\log x/j = \log x - \log j$

$$J_{31}(x) = \prod_{p|N} \left(1 - \frac{1}{p}\right) \prod{}' \left(1 - \frac{1}{p^2}\right) (J_{41}(y) \log x - J_{42}(y))$$
$$+ 0\left((\log\log 3Nx) \log x + \frac{\tau(N) \log x}{x^{1/2}}\right), \tag{5}$$

where

$$J_{41}(y) = \sum_{j\leqslant y}' \frac{\mu^2(j)}{j} \prod_{p|j} \left(\frac{p}{p+1}\right)$$

$$J_{42}(y) = \sum_{j\leqslant y}' \frac{\mu^2(j)}{j} \log j \prod_{p|j} \left(\frac{p}{p+1}\right).$$

By using (2) above with $g(p) = -1/(1+p)$

$$J_{41}(y) = \prod_{p|N} \left(1 - \frac{1}{p}\right) \prod{}' \left(1 - \frac{1}{p^2}\right)\left(1 - \frac{1}{(p+1)^2}\right) \log y$$

$$+ 0\left(\log\log 3N + \frac{\tau(N)}{y}\right). \tag{6}$$

Using Lemma 2.2 with $C = 1$ and $f(p) = p/(p+1)$

$$J_{42}(y) = \prod_{j\leqslant y}' \left(1 - \frac{1}{(p+1)^2}\sum{}' \frac{\mu^2(j)}{j} \log j + 0(\log y).\right.$$

The sum above is

$$J_{51}(y) = \sum_{j\leqslant y}' \frac{\mu^2(j)}{j} \log j = \sum_{j\leqslant y}' \frac{\log j}{j} \sum_{i^2|j} \mu(i)$$

$$= \sum_{i^2k\leqslant y}' \frac{\log i^2k}{i^2k} \mu(i)$$

$$= \sum_{k\leqslant y}' \frac{\log k}{k} \sum_{i^2\leqslant y/k}' \frac{\mu(i)}{i^2} + 2\sum_{k\leqslant y}' \frac{1}{k} \sum_{i^2\leqslant y/k}' \frac{\mu(i)\log i}{i^2}.$$

Proceeding much as in (3.3) with the first sum and using

$$\sum_{i\leqslant\infty}' \frac{\log i}{i^2} < \infty$$

in the second sum

$$J_{51}(y) = \prod{}' \left(1 - \frac{1}{p^2}\right) \sum_{k\leqslant y}' \frac{\log k}{k} + 0(\log y)$$

Hence

$$J_{42}(y) = \prod{}' \left(1 - \frac{1}{p^2}\right)\left(1 - \frac{1}{(p+1)^2}\right) \sum_{k\leqslant y}' \frac{\log k}{k} + 0(\log y).$$

But

$$\sum_{k\leqslant y}{}' \frac{\log k}{k} = \sum_{k\leqslant y} \frac{\log k}{k} \sum_{\substack{i|k\\ i|N}} \mu(i)$$

$$= \sum_{i|N} \mu(i) \sum_{\substack{i|k\\ k\leqslant y}} \frac{\log k}{k}$$

$$= \sum_{i|N} \frac{\mu(i)}{i} \sum_{j\leqslant y/i} \left(\frac{\log i}{j} + \frac{\log j}{j}\right).$$

By using (2.21) on $\sum 1/j$ and $\sum \log j/j$

$$\sum_{k\leqslant y}{}' \frac{\log k}{k} = \sum_{i|N} \frac{\mu(i)}{i} \log i \left(\log \frac{y}{i} + \gamma + 0\left(\frac{i}{y}\right)\right)$$

$$+ \sum_{i|N} \frac{\mu(i)}{i} \left(\frac{1}{2} \log^2 \frac{y}{i} + C_1 + 0\left(\frac{i}{y} \log \frac{y}{i}\right)\right)$$

$$= \frac{1}{2} \log^2 y \prod_{p|N} \left(1 - \frac{1}{p}\right) - \frac{1}{2} \sum_{i|N} \frac{\mu(i)}{i} \log^2 i$$

$$+ \gamma \sum_{i|N} \frac{\mu(i)}{i} \log i + C_1 \prod_{p|N} \left(1 - \frac{1}{p}\right)$$

$$+ 0\left(\frac{\tau(N)}{y} (\log N + \log y)\right).$$

By using Lemmas 4.1 and 4.2

$$\sum_{k\leqslant y}{}' \frac{\log k}{k} = \frac{1}{2} \log^2 y \prod_{p|N} \left(1 - \frac{1}{p}\right) + 0\left\{(\log\log 3N)^2 + \frac{\tau(N)}{y} \log Ny\right\}.$$

Thus

$$J_{42}(y) = \frac{1}{2} \log^2 y \prod_{p|N} \left(1 - \frac{1}{p}\right) \prod{}' \left(1 - \frac{1}{p^2}\right) \left(1 - \frac{1}{(p+1)^2}\right)$$

$$+ 0\left((\log\log 3N)^2 + \log y + \frac{\tau(N)}{y} \log Ny\right).$$

Hence this and (6) in (5) with $y = x^{1/2}$ gives

$$J_{31}(x) = \frac{3}{8} \prod_{p|N} \left(1 - \frac{1}{p}\right)^2 \prod{}' \left(1 - \frac{1}{p^2}\right)^2 \left(1 - \frac{1}{(p+1)^2}\right) \log^2 x$$

$$+ 0\left(\log x \log\log 3Nx + \frac{\tau(N) \log xN}{x^{1/2}} + (\log\log 3N)^2\right). \qquad (7)$$

Turning now to $J_{32}(x)$ below (4) and using (2) in the inner sum with x replaced by y and N by Nj

$$\sum_{\substack{k\leqslant y\\(k,Nj)=1}} \frac{\mu^2(k)}{k} = \prod\left(1-\frac{1}{p^2}\right)\prod_{p|Nj}\left(\frac{p}{p+1}\right)\log y$$
$$+ 0\left(\log\log 3Nj + \frac{\tau(N)\,\tau(j)}{y}\right).$$

Hence

$$J_{32}(x) = \prod_{p|N}\left(1-\frac{1}{p}\right)\prod{}'\left(1-\frac{1}{p^2}\right)\log y \sum_{j\leqslant y}{}' \frac{\mu^2(j)}{j}\prod_{p|j}\frac{p}{p+1}$$
$$+ 0\left(\log\log 3Nx\log x + \frac{\tau(N)}{y}\log^2 y\right).$$

The sum is $J_{41}(y)$ and hence by (6)

$$J_{32}(x) = \prod_{p|N}\left(1-\frac{1}{p}\right)^2\prod{}'\left(1-\frac{1}{p^2}\right)^2\left(1-\frac{1}{(p+1)^2}\right)\log^2 y$$
$$+ 0\left\{(\log\log 3Nx)\log x + \frac{\tau(N)}{x^{1/2}}\log^2 x\right\}.$$

Now using this and (7) in (4)

$$J_{21}(x) = \frac{1}{2}\log^2 x\prod_{p|N}\left(1-\frac{1}{p}\right)^2\prod{}'\left(1-\frac{1}{p^2}\right)^2\left(1-\frac{1}{(p+1)^2}\right)^2$$
$$+ 0\left\{\left(\log\log 3Nx + \frac{\tau(N)\log Nx}{x^{1/2}}\right)\log x + (\log\log 3N)^2\right\}.$$

This modifies $J_{21}(x)$ at the bottom of p. 641 by the term involving $\tau(N)$ and returning to (3.5) the proof is complete.

ADDENDUM. To obtain an upper bound in the number of solutions of $p_1 + p_2 = M$ even, and p_1 and p_2 prime, let N be square-free and contain those primes which are factors of M. One encounters, reference [2, Chap. 6] of [1], the sum

$$J_3(x) = \sum_{d|N}\frac{1}{\phi(d)}J_2\left(\frac{x}{d}\right), \tag{8}$$

where $\phi(n)$ is Euler's function and

$$J_2(x) = \frac{C}{2}\log^2 x + I, \qquad C = \prod_{p|N}\left(1-\frac{1}{p}\right)^2\prod{}'\left(1+\frac{1}{p(p-2)}\right)$$

and

$$I = 0\left\{\left(\log\log 3Nx + \frac{\tau(N)\log Nx}{x^{1/2}}\right)\log x + (\log\log 3N)^2\right\}.$$

Hence, taking account of the case $x < d \leqslant N$,

$$J_2\left(\frac{x}{d}\right) = \frac{C}{2}\log^2\frac{x}{d} + I + 0\left(\frac{\tau(N)\,d\log^2 x}{x}\right)$$

since $\log N \leqslant 4\log x$ because $x > N^{1/4}$ in applications. Hence

$$J_2\left(\frac{x}{d}\right) = \frac{C}{2}\log^2 x + I + 0\left(\frac{\tau(N)\,d\log^2 x}{x} + \log x\log d\right). \tag{9}$$

But

$$\sum_{d|N}\frac{1}{\phi(d)} = \prod_{p|N}\left(1 + \frac{1}{p-1}\right) = \frac{N}{\phi(N)} \tag{10}$$

and so

$$\sum_{d|N}\frac{1}{\phi(d)} = \prod_{p|N}\left(\frac{p^2}{p^2-1}\right)\prod_{p|N}\left(1+\frac{1}{p}\right) = 0\left(\exp\sum_{p|N}\frac{1}{p}\right) = 0(\log\log 3N), \tag{11}$$

where use is made of a result similar to (4.6) of [1]. Also

$$\sum_{d|N}\frac{\log d}{\phi(d)} = \sum_{d|N}\frac{1}{\phi(d)}\sum_{p|d}\log p = \sum_{p|N}\frac{\log p}{p-1}\sum_{j|N/p}\frac{1}{\phi(j)}$$

$$= \frac{N}{\phi(N)}\sum_{p|N}\frac{\log p}{p-1}\frac{p-1}{p} = 0\{(\log\log 3N)^2\}, \tag{12}$$

where use is made of (4.6). Finally

$$\sum_{d|N}\frac{d}{\phi(d)} = \prod_{p|N}\left(1+\frac{p}{p-1}\right) = \tau(N)\prod_{p|N}\left(\frac{p-\frac{1}{2}}{p-1}\right)$$

$$\leqslant \tau(N)\prod_{p|N}\left(1+\frac{1}{p}\right)$$

$$= 0(\tau(N)\log\log 3N). \tag{13}$$

By combining (9)-(13) in (8)

$$J_3(x) = \frac{1}{2}\log^2 x \prod_{p|N}\left(1 - \frac{1}{p}\right) \prod{}' \left(1 + \frac{1}{p(p-2)}\right) + \tilde{I}$$

$$\tilde{I} = (\log\log 3N)\, 0 \left\{\left(\log\log 3Nx + \frac{\tau(N)\log xN}{x^{1/2}}\right)\log x \right.$$
$$\left. + (\log\log 3N)^2 + \frac{\tau^2(N)\log^2 x}{x}\right\}.$$

The error term for $J_3(x)$ using $x > N^{1/4}$ can be replaced by

$$\tilde{I} = 0\{(\log\log x)^2 \log x\}.$$

Reference

1. N. Levinson. Summing certain number theoretic series arising in the sieve. *J. Math. Anal. Appl.* **22** (1968), 631–645.